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A VARIATIONAL METHOD OF SOLVING AN ELASTIC-PLASTIC PROBLEM FOR A BODY WITH A CIRCULAR HOLE*

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An approach based on the theory of variational inequalities and a generalized plastic analogy for the solution of the elastic-plastic problem (EPP) concerning the state of stress of a body weakened by a circular hole without the assumption regarding total enclosure of the hole by a plastic zone is proposed. The Haar-Karman hypothesis or an equivalent assertion is not used here. Generalizations are given to the case of a plastic inhomogeneous body and for the utilization of an exponential flow condition. Examples are considered and a simple method is proposed for estimating the plastic zone dimensions.

It was assumed in the well-known solution given by Galin /1/ of the EPP on the biaxial tension of a plane with a circular hole and its generalizations /2-5/ that the plastic domain completely encloses the hole. The majority of existing solutions have been obtained for the stress concentration around a hole in an infinite domain.

Let us consider the problem of the plane strain of a body Ω with smooth outer contour L and a circular hole C of radius a (Fig.1). Near the outer boundary the medium under the loads acting on the body is in an elastic state. We shall also assume that if the plastic zone does not enclose the hole, then all its connected subdomains lie within appropriate characteristic triangles such that, as in the case of total enclosure, the stresses in the plastically homogeneous zone D^p are described by the relationships (tensile conditions)

$$\sigma_{rr}^p = 2\tau_s \ln (r/a), \quad \sigma_{\vartheta\vartheta}^p = 2\tau_s [1 + \ln (r/a)], \quad \sigma_{r\vartheta}^p = 0 \tag{1}$$

where r, ϑ are polar coordinates connected to the centre of the hole and τ_s is the plasticity limit.

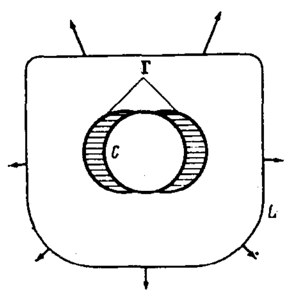

Fig.1

It is convenient to formulate a statically determinable EPP in the terminology of the Airy stress function /1-3/: it is required to find the function $u(x, y)$ which satisfies the biharmonic equation in the elastic zone D^e

$$\Delta^2 u = 0 \tag{2}$$

and the condition

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 < 4\tau_s^2 \tag{3}$$

and satisfies the following equation in the plastic zone D^p:

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 4\tau_s^2 \tag{4}$$

The conditions

$$u|_L = f(l), \quad \left.\frac{\partial u}{\partial n}\right|_L = f_1(l), \quad u|_C = 0, \quad \left.\frac{\partial u}{\partial n}\right|_C = 0 \tag{5}$$

*Prikl.Matem.Mekhan., 52, 1, 126-132, 1988 are specified on the body boundary.

The function $u(x, y)$, together with its derivatives to second order inclusive, is continuous on the elastic-plastic boundary Γ, unknown in advance, and corresponds to a continuous stress field.

Let $u^e(x, y)$ be the solution of (2) with the boundary conditions (5). The corresponding purely elastic stress field is smooth within the body.

Assertion. If $u(x, y)$ is the solution of the EPP (2)-(5), then the function $U(x, y) = u - u^e$ minimizes the functional

$$J(v) = \iint_\Omega (\Delta v)^2 \, d\Omega \tag{6}$$

in the set of allowable functions K:

$$v(x, y) \in W_2^2(\Omega), \ v|_{\partial\Omega} = 0, \ \partial v/\partial n|_{\partial\Omega} = 0 \tag{7}$$

$$\left[\frac{\partial^2}{\partial x^2}(v + u^e) - \frac{\partial^2}{\partial y^2}(v + u^e)\right]^2 + 4\left[\frac{\partial^2 (v + u^e)}{\partial x \partial y}\right]^2 \leqslant 4\tau_s^2 \tag{8}$$

Proof. In the case of homogeneous plastic properties of a material, the function $u(x, y)$ in the plastic zone is identical with the biharmonic function $u^p(r)$ corresponding to the stress field (1) so that $U(x, y)$ satisfies (2) in each of the subdomains D^e and D^p.

Let us obtain a variational inequality for $U(x, y)$. We multiply the biharmonic operator of U by the function $(v - U)$, where $v(x, y)$ is an arbitrary element of the convex set K. We integrate this expression in each of the subdomains D^e and D^p. After applying Green's formula and summation of the integrals over both zones the following relationship is obtained:

$$a(U, v - U) = \iint_\Omega \Delta U \cdot \Delta(v - U) \, d\Omega = -\int_\Gamma (v - U)\,\delta(l)\, dl + \tag{9}$$

$$\int_{\partial\Omega} \left\{\Delta U \frac{\partial}{\partial n}(v - U) - (v - U)\frac{\partial}{\partial n}\Delta U\right\} dl$$

$$\delta = \left[\frac{\partial^3 U}{\partial n^3}\right] = \frac{\partial^3 u}{\partial n^3}\bigg|_{\Gamma^p} - \frac{\partial^3 u}{\partial n^3}\bigg|_{\Gamma^e} \tag{10}$$

Here δ is the jump in the normal derivative of the shear stress $\sigma_{tt} = \partial^2 u/\partial n^2$ during passage from the elastic into the plastic zone (the normal n is external relative to D^p). The fact, established in /6/, of discontinuity of the derivative $\partial\sigma_{tt}/\partial n$ almost everywhere on the elastic-plastic boundary Γ is used here. The remaining three derivatives of $U(x, y)$ in the local coordinates (t, n) are continuous in the neighbourhood of Γ.

For all the functions $v \in K$ the integral over the boundary of the body $\partial\Omega$ on the right-hand side of (9) vanishes so that it is sufficient to show constancy of the sign of the remaining integral over the elastic-plastic boundary Γ. The positivity of the jump of the derivative (10) for the solution of the EPP results from the relationship /6/ (taking the condition $\sigma_{tt} > \sigma_{nn}$ into account)

$$\left[\frac{\partial\sigma_{tt}}{\partial n}\right] = \frac{4\tau_m}{\sigma_{nn} - \sigma_{tt}}\left(\frac{\partial\tau_m}{\partial n}\right)_{\Gamma^e}, \quad \tau_m = \frac{1}{2}\sqrt{(\sigma_{nn} - \sigma_{tt})^2 + 4\sigma_{nt}^2}$$

Indeed, the maximum shear stress is $\tau_m = \tau_s$ (4) everywhere in the plastic domain while the inequality $\tau_m < \tau_s$, is satisfied in adjacent points of the domain D^e so that $\delta(l) \geqslant 0$.

Furthermore, to be specific the case of tensile stresses is examined. We will show that for an arbitrary function $v(x, y)$ from the set K the inequality

$$v(x, y) \leqslant U(x, y) = u^p(r) - u^e(r, \vartheta) = F(r, \vartheta) \tag{12}$$

is satisfied in this case in the whole plastic zone up to the boundary Γ.

Indeed, the inequality (8) that yields the set K can be rewritten in a polar coordinate system in the form of the relationships

$$\tau_{\vartheta\vartheta} - \tau_{rr} = M[v + u^e] = 2\tau_s - \Phi(r, \vartheta), \quad \Phi \geqslant 0 \tag{13}$$

$$M = \partial^2/\partial r^2 - r^{-1}\partial/\partial r - r^{-2}\partial^2/\partial\vartheta^2$$

$$\tau_{rr} = \frac{1}{r}\frac{\partial(v + u^e)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(v + u^e)}{\partial\vartheta^2}, \quad \tau_{\vartheta\vartheta} = \frac{\partial^2(v + u^e)}{\partial r^2}$$

Here τ_{rr}, $\tau_{\theta\theta}$ are the stress components corresponding to the "trial" Airy function $v + u^e$, where the allowable stress functions satisfy the condition $\tau_{\theta\theta} > \tau_{rr}$ in D^p for the tension case, as does the solution of (1).

The equality

$$M[u] = M[u^p] = 2\tau_s \tag{14}$$

is satisfied for a real stress field in the plastic zone.

Therefore, the function $w = u^p - u^e - v$ satisfies the equation with conditions on the hole outline

$$M[w] = \Phi(r, \theta) \geqslant 0;\ w|_{r=a} = \partial w/\partial r\,|_{r=a} = 0 \tag{15}$$

where $\Phi(r, \theta)$ is a certain non-negative function.

The solution of problem (15) is conveniently analysed in the variables $\xi = \ln r + \theta$, $\eta = \ln r - \theta$ where we obtain after chaning to them

$$\frac{\partial^2 u}{\partial\xi\partial\eta} - \frac{1}{2}\frac{\partial u}{\partial\xi} - \frac{1}{2}\frac{\partial u}{\partial\eta} = \frac{1}{4}e^{\xi+\eta}\Phi(\xi, \eta) = \Psi(\xi, \eta) \geqslant 0 \tag{16}$$
$$u = \partial u/\partial\xi = \partial u/\partial\eta = 0 \quad \text{for} \quad \xi + \eta = 2\ln a$$

The solution of this Cauchy problem by the Riemann method in the domain $\xi + \eta > 2\ln a$ is represented in the form

$$u(\xi_0, \eta_0) = \iint\limits_{\xi\leqslant\xi_0,\ \eta\leqslant\eta_0} V(\xi, \eta; \xi_0, \eta_0)\,\Psi(\xi, \eta)\,d\xi\,d\eta$$

Here the Riemann-Green function $V(\xi, \eta; \xi_0, \eta_0)$ is the solution of the Goursat problem for the adjoint equation

$$\frac{\partial^2 V}{\partial\xi\partial\eta} + \frac{1}{2}\frac{\partial V}{\partial\xi} + \frac{1}{2}\frac{\partial V}{\partial\eta} = 0$$
$$V|_{\xi=\xi_0} = \varphi_1(\eta) = \exp[(\eta_0 - \eta)/2],\quad V|_{\eta=\eta_0} = \varphi_2(\xi) = \exp[(\xi_0 - \xi)/2]$$

Here in the case under consideration

$$\varphi_1(\eta) > 0,\ \varphi_2(\xi) > 0,\ \varphi_1'(\eta) < 0,\ \varphi_2'(\xi) < 0$$

The Riemann function is constructed by successive approximations /7/. Introducing the notation

$$w^{(1)}(\xi, \eta) = \partial V/\partial\xi,\ w^{(2)}(\xi, \eta) = \partial V/\partial\eta$$

we rewrite the solution in the form

$$V = V_0 + \sum_{n=1}^{\infty}(V_n - V_{n-1})$$
$$w^{(1)} = w_0^{(1)} + \sum_{n=1}^{\infty}[w_n^{(1)} - w_{n-1}^{(1)}],\quad w^{(2)} = w_0^{(2)} + \sum_{n=1}^{\infty}[w_n^{(2)} - w_{n-1}^{(2)}]$$

where the subscript indicates the number of the approximation and

$$V_0 = \varphi_2(\xi) > 0,\ w_0^{(1)} = \varphi_2'(\xi) < 0,\ w_0^{(2)} = \varphi_1'(\eta) < 0$$

The constancy of the signs of terms of the series for the Riemann function and its derivatives

$$V_n - V_{n-1} \geqslant 0,\quad w_n^{(1)} - w_{n-1}^{(1)} \leqslant 0,\quad w_n^{(2)} - w_{n-1}^{(2)} \leqslant 0$$

is proved by induction using iteration formulas /7/.

Therefore, the Riemann function for the hyperbolic operator M is non-negative. Hence, the solution of the Cauchy problem (15) is also non-negative for an arbitrary non-negative right side Φ. Consequently, we have $v \leqslant U$ in the plastic zone D^p.

Together with the inequality for the jump δ this yields the non-negativity of the right-hand side of (9), i.e., the variational inequality is satisfied

$$a(U, v - U) \geqslant 0 \tag{17}$$

The signs of the right-hand sides in (1), (11), and (15) change to the opposite for the case of compressive stresses so that $\delta < 0$ and the trial function is $v(x, y) \geqslant U = -F(r, \theta)$, and the inequality (17) is also satisfied. Since the set K is convex and closed while the bilinear form $a(v_1, v_2)$ is positive-definite in the space (7), the existence of a unique element $U \in K$ satisfying inequality (17) for all $v \in K$ follows from the theory of variational inequalities /8/ and this EPP solution supplies the minimum of the functional (6) in the set K. The sufficiency is proved by a standard method /8/. It should be noted that the variational inequality (17) is obtained without any assumptions regarding the equation of state of the plastic body (of the type of the Haar-Karman hypothesis, the Mises maximum principle, etc., which were used in /8, 9/).

In general, after constructing the solution of problem (6)-(8), it is necessary to confirm that the characteristics of the ideal plasticity equations emerging from the hole outline intersect the elastic-plastic boundary just once.

The proof presented for the variational principle is easily extended to the case of inhomogeneous plastic properties of the material round the hole when the plasticity limit, as in /10/, depends on the distance to the hole outline $\tau_s = \tau_s(r)$. Suppose the following additional requirement is satisfied:

$$\Delta(\sigma_{rr}{}^p + \sigma_{\theta\theta}{}^p) = \Delta^2 u^p = 2\tau_s''(r) + 6\tau_s'(r)/r \leqslant 0 \tag{18}$$

Then inequality (17) also holds. Indeed, in this case a non-negative term

$$\iint\limits_{D^p} (v - U)\,\Delta^2 u^p\,dS$$

is added on the right-hand side of relationship (9).

On the basis of an interpretation of the exact solution be obtained /1/, Galin proposed /11/ a platelike analogy for the EPP regarding the biaxial tension of a plane with a hole. By using the inequalities obtained above, the generalized platelike analogy can be given a strict foundation. It follows from the inequality $\delta \geqslant 0$ that the elastic-plastic Airy function is $u(x, y) \leqslant u^p(r)$ in D^e in the neighbourhood of the boundary Γ and we obtain from the relationship (15) for the function $w = u^p - u$ that the inequality $u \leqslant u^p(r)$ will be satisfied for domains Ω in which the field of plastic characteristics allows of continuation ot the outer contour L. Then the solution of the initial problem (2)-(5) can be replaced by the solution of the problem of the bending of a plate of planform Ω (u is its deflection) for the given boundary conditions (5) and the constraint on the deflection

$$u(x, y) \leqslant u^p(r) \tag{19}$$

Indeed, the EPP solution $u(x, y)$ in the "contact" zone D^p coincides with the "plastic" surface $u = u^p(r)$ and the "reaction" of this surface is directed towards the plate (for $\Delta^2 u^p \leqslant 0$ and taking account of the inequality $\delta \geqslant 0$ for the "transverse forces" concentrated along the line Γ). In the domain D^e where the plate is free of a normal load it lies below the surface $u = u^p(r)$. As in the EPP, the derivatives to second order inclusive are continuous on the boundary Γ (there are no concentrated moments).

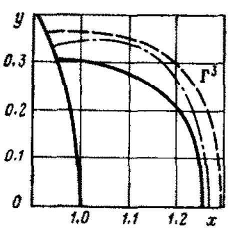

Fig.2

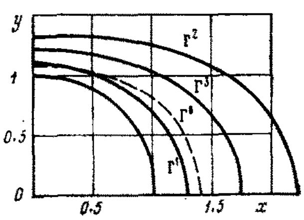

Fig.3

Since the boundary conditions (5) for non-zero f and f_1 do not allow of a suitable physical realization it is **convenient** to change to the problem with zero boundary conditions $\zeta = u - u^e$. We therefore arrive at the problem of the pressure of a stamp with the profile $F = u^p(r) - u^e(r, \theta)$ on a doubly-connected plate clamped along the contour $\partial\Omega$. The solution of this problem satisfies the variational inequality

$$a(\zeta, \zeta' - \zeta) \geqslant 0, \quad \zeta, \zeta' \in K_1 \tag{20}$$
$$K_1 = \{\zeta' \mid \zeta' \in W_2^{\circ 2}(\Omega),\ \zeta'(x, y) \leqslant F(x, y)\}$$

where K_1 is a convex closed set of functions from (7).

As usual, the solution of the variational inequality (20) can be obtained by minimizing the functional $a(\zeta', \zeta')$ on K_1

$$J[U] = \inf\nolimits_{K_1} a(\zeta', \zeta') \tag{21}$$

The numerical realization of the variational problem (21) is substantially simpler than minimization of the functional (6) in the initial set K. It has been shown /12/ that the solution of problem (21) for $\Delta^2 F \leqslant 0$ possesses sufficient smoothness $(U \in W_2^3(\Omega), U \in C^2(\Omega))$ and a domain of no contact (9n the analogy D^e) is connected for this solution.

As an illustration of the utilization of the variational approach, the problem is considered for a wide ring $1 \leqslant r \leqslant 6.5$ for boundary conditions on the outer circle taken by solving the elastic problem about uniaxial tension of a plane with a hole by a load $\sigma_{xx} = 0$, $\sigma_{yy} = 0.833\tau_s$. Its solution was constructed by the local variations method /13/ and was compared with the approximate EPP solution for the tension of a plane with a hole /14/ (the plastic zone does not enclose the hole). The problem was solved for a quadrant of the ring with the partitioning along the angle ϑ with the step $\pi/60$, and along the radius with the variable step: 0.05 for $r \leqslant 2.5$ and 0.1 for $r > 2.5$. Results are presented in Fig.2 for the elastic-plastic boundary (dash-dot), while the approximate solution /14/ is shown there by the solid line.

We also note the simple "external" estimate for the plastic zone which results from the platelike analogy. Since the area of contact between the stamp and the plate is contained completely in the domain

$$G_2 = \{(r, \vartheta) \in \Omega : F(r, \vartheta) = u^p(r) - u^e(r, \vartheta) \leqslant 0\}$$

the unknown elastic-plastic boundary lies within this domain with the boundary Γ^2. On the other hand, it is known that an approximate estimate can be given for the plasticity domain in terms of satisfying condition (3) for the purely elastic solution

$$D^p \approx G_1 : \left\{(x, y) \in \Omega : \left(\frac{\partial^2 u^e}{\partial x^2} - \frac{\partial^2 u^e}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 u^e}{\partial x \partial y}\right)^2 \geqslant 4\tau_s^2\right\}$$

where this estimate is mainly "internal". Suppose $\Gamma^2 : \{r = R_2(\vartheta)\}$ and $\Gamma^1 : \{r = R_1(\vartheta)\}$ are curves corresponding to the equality conditions that approximate the elastic-plastic boundaries in these estimate. Then it is natural to take the curve $\Gamma^3 : \{r = [R_1(\vartheta) + R_2(\vartheta)]/2\}$ as the appropriate approximation of the elastic-plastic boundary.

Fig.3 shows the boundary lines $\Gamma^1, \Gamma^2, \Gamma^3$, constructed for the conditions of the Galin problem /1/ regarding the tension in a plane with a hole of unit radius for $\sigma_{xx} = 1.3\tau_s$; $\sigma_{yy} = 1.6\tau_s$. The elastic-plastic boundary corresponding to the exact solution is also given there. The dashed curves in Fig.2 represent the appropriate approximate curve Γ^3 constructed according to the elastic solution for a plane with a hole. It is seen that the approach proposed for determing the unknown boundary yields good results when the plastic domain is small; the true boundary Γ^0 here lies everywhere within the "middle" outline Γ^3.

The EPP is investigated analogously in the case when the exponential flow condition /3/

$$(\sigma_{\vartheta\vartheta} - \sigma_{rr})^2 + 4\sigma_{r\vartheta}^2 = 4k^2\{1 - \exp[-\sigma_0/k + (\sigma_{rr} + \sigma_{\vartheta\vartheta})/(2k)]\}^2 \tag{22}$$

is satisfied in the plastic zone.

Here k and σ_0 are positive constants with the dimensionality of stress. In the case of an infinite medium with a circular hole the problem with the plasticity condition (22) was examined in /3/ under the assumption of total enclosure of the hole by the plastic zone.

For a variational approach it is convenient to formulate this problem, like the preceding one, in terms of the Airy stress function. The inequality

$$Mu \geqslant Nu, \quad Nu = -2k\left[1 - \exp\left(-\frac{\sigma_0}{k} + \frac{\Delta u}{2k}\right)\right] \tag{23}$$

is satisfied here in the elastic zone for the compression conditions (the operator M is defined in (13)).

Furthermore, it is assumed that the inequality $\sigma_{\vartheta\vartheta} < \sigma_{rr}$ /3/ holds under compression conditions in the elastic, as well as in the plastic, zone. Condition (22) which is satisfied in the plastic zone D^p has the form $Mu = Nu$ in the notation used.

Unlike the problem with the classical plasticity condition, let us make the additional assumption that the stress tensor components τ_{rr} and $\tau_{\vartheta\vartheta}$ found from the solution of the EPP on the outer contour L satisfy the condition

$$\tau_{rr} + \tau_{\vartheta\vartheta} > \sigma_{rr}^A + \sigma_{\vartheta\vartheta}^A \tag{24}$$

where $\sigma_{rr}^A, \sigma_{\vartheta\vartheta}^A$ and the "plastic" stresses (this assumption is satisfied for the exact solution in /3/).

Assertion. The inequality

$$u(x, y) \geqslant u_p^A(x, y) = \chi^A(r) \tag{25}$$

is satisfied when condition (24) is satisfied everywhere in the elastic zone D^e, where χ^A is the "plastic" Airy function for a body with flow condition (22).

Proof. By the definition of the functions χ^A it satisfies the relationship $M\chi^A = N\chi^A$ in Ω. Subtracting the corresponding parts of (23) from both sides of this equality, we obtain in the elastic zone

$$M(\chi^A - u) < N\chi^A - Nu \tag{26}$$

Let us consider the right side of relationship (26)

$$2k\left[\exp\left(-\frac{\sigma_0}{2k}+\frac{\Delta\chi^A}{2k}\right)-\exp\left(-\frac{\sigma_0}{k}+\frac{\Delta u}{2k}\right)\right]$$

Since the functions χ^A and u are biharmonic in the domain D^e /3/, it follows from (24) and the condition of agreement of χ^A and u on Γ up to the second derivatives, that $\Delta\chi^A < \Delta u$ in D^e. It hence follows that $N\chi^A \leqslant Nu$ in D^e, and therefore, in conformity with (26), the inequality $M(\chi^A - u) < 0$ is satisfied in the elastic domain. It has been shown above that inequality (25) follows from this.

Repeating the discussion elucidated above, we arrive at the equivalence of the EPP under consideration with the contact problem on the bending of a plate with planform Ω by a stamp $z = F^A = \chi^A(r) - u^e(x, y)$, which reduces, in turn, to the extremal problem (21) on a convex set

$$K_1' = \{\zeta' \mid \zeta' \in \overset{\circ}{W}{}_2^{2}(\Omega),\quad \zeta'(x, y) \geqslant F^A(x, y)\}$$

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